TOPOLOGICAL RECURSION AND THE QUANTUM CURVE FOR MONOTONE HURWITZ NUMBERS

Embedded Graphs
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Take a permutation and count the number of ways to express it as a product of a fixed number of transpositions — you have calculated a Hurwitz number. By adding a mild constraint on such factorisations, one obtains the notion of a monotone Hurwitz number. We have recently shown that the monotone Hurwitz problem fits into the so-called topological recursion/quantum curve paradigm. This talk will attempt to explain what the previous sentence means.

Simple Hurwitz numbers

Hurwitz numbers count the number of ways to express a permutation as a product of transpositions.

Definition

Let $H_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ be $\frac{1}{|\boldsymbol{\mu}|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of transpositions in $S_{|\boldsymbol{\mu}|}$ such that

- $m = 2g - 2 + n + |\boldsymbol{\mu}|$;
- $\sigma_1 \sigma_2 \cdots \sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \ldots, \mu_n$; and
- $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$ is transitive.

Fact

Hurwitz numbers equivalently count

- branched covers of $\mathbb{CP}^1$ with respect to ramification over ∞;
- edge-labelled embedded graphs with respect to winding number.

Monotone Hurwitz numbers

For monotone Hurwitz numbers, we add a mild constraint.

Definition

Let $\vec{H}_{g,n}(\mu_1, \mu_2, \ldots, \mu_n)$ be $\frac{1}{|\boldsymbol{\mu}|!}$ multiplied by the number of tuples $(\sigma_1, \sigma_2, \ldots, \sigma_m)$ of transpositions in $S_{|\boldsymbol{\mu}|}$ such that

- $m = 2g - 2 + n + |\boldsymbol{\mu}|$;
- $\sigma_1 \sigma_2 \cdots \sigma_m$ has labelled cycles of lengths $\mu_1, \mu_2, \ldots, \mu_n$;
- $\langle \sigma_1, \sigma_2, \ldots, \sigma_m \rangle$ is transitive; and
- if $\sigma_i = (a_i \; b_i)$ with $a_i < b_i$, then $b_1 \leq b_2 \leq \cdots \leq b_m$.

Why monotone?

Monotone Hurwitz numbers are natural from the viewpoint of

- matrix models (HCIZ integral);
- representation theory (Jucys–Murphy elements); and
- integrability (Toda tau-functions).

Example calculation

Take $(g,n)=(0,2)$ and $\boldsymbol{\mu}=(1,2)$, so $m=2g-2+n+|\boldsymbol{\mu}|=3$.

There are 27 products of 3 transpositions in S_3 and 24 are transitive.

$$
\begin{array}{llll}
(1\ 2)\circ(1\ 2)\circ(1\ 3) & (1\ 2)\circ(1\ 3)\circ(2\ 3) & (1\ 3)\circ(1\ 3)\circ(2\ 3) & (2\ 3)\circ(1\ 3)\circ(1\ 3) \\
(1\ 2)\circ(1\ 2)\circ(2\ 3) & (1\ 2)\circ(2\ 3)\circ(1\ 3) & (1\ 3)\circ(2\ 3)\circ(1\ 3) & (2\ 3)\circ(1\ 3)\circ(2\ 3) \\
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(1\ 3)\circ(1\ 2)\circ(1\ 2) & (1\ 3)\circ(1\ 3)\circ(1\ 2) & (2\ 3)\circ(1\ 2)\circ(1\ 3) & (2\ 3)\circ(2\ 3)\circ(1\ 2)
\end{array}
$$

All 24 products produce cycle type $(1,2)$, so $H_{0,2}(1,2)=\frac{24}{3!}=4$.

Only the first 12 products are monotone, so $\vec{H}_{0,2}(1,2)=\frac{12}{3!}=2$.

Polynomiality

Theorem (Ekedahl–Lando–Shapiro–Vainshtein, 2001, and Goulden–Guay-Paquet–Novak, 2013)

There are polynomials $P_{g,n}$ and $\vec{P}_{g,n}$ such that

- $H_{g,n}(\mu_1, \ldots, \mu_n) = m! \times \prod \frac{\mu_i^{\mu_i}}{\mu_i!} \times P_{g,n}(\mu_1, \ldots, \mu_n)$
- $\vec{H}_{g,n}(\mu_1, \ldots, \mu_n) = \prod \binom{2\mu_i}{\mu_i} \times \vec{P}_{g,n}(\mu_1, \ldots, \mu_n)$.

For example,

- $P_{1,2}(\mu_1, \mu_2) = \frac{1}{24}(\mu_1^2 + \mu_2^2 + \mu_1\mu_2 - \mu_1 - \mu_2)$
- $\vec{P}_{1,2}(\mu_1, \mu_2) = \frac{1}{12}(2\mu_1^2 + 2\mu_2^2 + 2\mu_1\mu_2 - \mu_1 - \mu_2 - 1)$.

In fact, $[\mu_1^{a_1} \cdots \mu_n^{a_n}]\, P_{g,n}(\mu_1, \ldots, \mu_n) = \pm \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_{3g-3+n-|\boldsymbol{a}|}$.

Question

Do the coefficients of $\vec{P}_{g,n}$ have geometric meaning?

Cut-and-join recursion

(Monotone) Hurwitz numbers of type (g, n) can be calculated from those of types

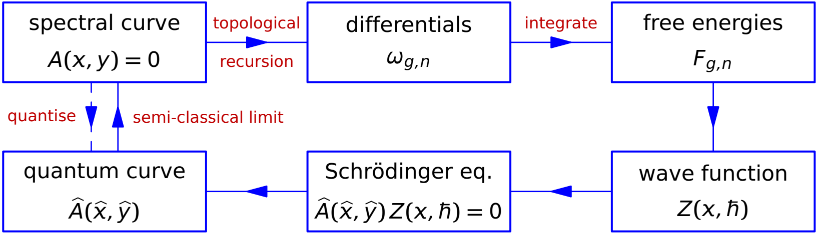
- $(g, n-1)$
- $(g-1, n+1)$
- $(g_1, n_1) \times (g_2, n_2)$ for $\begin{cases} g_1 + g_2 = g \\ n_1 + n_2 = n + 1. \end{cases}$

For example,

$$\mu_1 \vec{H}_{1,2}(\mu_1, \mu_2) = (\mu_1 + \mu_2) \vec{H}_{1,1}(\mu_1 + \mu_2) + \sum_{\alpha+\beta=\mu_1} \alpha\beta \vec{H}_{0,3}(\alpha, \beta, \mu_2)$$

$$+ \sum_{\alpha+\beta=\mu_1} \alpha\beta \left[\vec{H}_{0,1}(\alpha) \vec{H}_{1,2}(\beta, \mu_2) + \vec{H}_{1,1}(\alpha) \vec{H}_{0,2}(\beta, \mu_2) \right].$$

Topological recursion and quantum curves



- Topological recursion (Chekhov–Eynard–Orantin):

$$\omega_{g,n}(z_S) = \sum_{\alpha} \operatorname*{Res}_{z=\alpha} K(z_1,z) \left[\omega_{g-1,n+1}(z,\overline{z},z_{S\setminus\{1\}}) + \sum^{\circ}_{\substack{g_1+g_2=g \\ I\sqcup J=S\setminus\{1\}}} \omega_{g_1,|I|+1}(z,z_I)\,\omega_{g_2,|J|+1}(\overline{z},z_J) \right]$$

- Wave function: $Z(x,\hbar) = \exp\left[\sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x,\ldots,x) \right]$
- Polarisation: $\widehat{x}=x$ and $\widehat{y}=-\hbar\frac{\partial}{\partial x}$, which imply $[\widehat{x},\widehat{y}]=\hbar$

Results

This is joint work with A. Dyer and D. Mathews (`arXiv:1408.3992`).

- The spectral curve $A(x, y) = xy^2 + y + 1 = 0$ yields

$$F_{g,n}(x_1, \ldots, x_n) = \sum_{\boldsymbol{\mu}}^{\infty} \vec{H}_{g,n}(\mu_1, \ldots, \mu_n)\, x_1^{\mu_1} \cdots x_n^{\mu_n}.$$

- The wave function satisfies

$$Z(x, \hbar) = 1 + \sum_{d=1}^{\infty} \sum_{m=0}^{\infty} \left\{ \begin{matrix} d+m-1 \\ d-1 \end{matrix} \right\} \frac{x^d\, \hbar^{m-d}}{d!}.$$

- The corresponding quantum curve is $\widehat{A}(\widehat{x}, \widehat{y}) = \widehat{x}\widehat{y}^2 + \widehat{y} + 1$, so

$$x\hbar^2 \frac{\partial^2 Z}{\partial x^2} - \hbar \frac{\partial Z}{\partial x} + Z = 0.$$

TR	**QC**	**PROBLEM**
N, DMSS	MS	Ribbon graph enumeration
AC, KZ	KZ, DoN	Dessin enumeration
EO	Z	Intersection theory on $\overline{M}_{g,n}$
EMS	Z, MSS	Simple Hurwitz numbers
DoLN, BHLM	MSS	Orbifold Hurwitz numbers
DoDM	DoDM	Monotone Hurwitz numbers
DOSS	DMNPS	Gromov–Witten theory of $\mathbb{CP}^1$
EO	Z	One-legged topological vertex
DM	DM	$SL(2,\mathbb{C})$ Hitchin fibrations
???	DoM	Hypermap enumeration
???	MSS	Spin Hurwitz numbers
EO	???	Weil–Petersson volumes
EO, FLZ	???	Gromov–Witten theory of toric CY3s
???	???	Coloured Jones polynomials of knots
???	???	Coloured HOMFLY polynomials of knots

AC Ambjørn, Chekhov. *The matrix model for dessins d'enfants* (2014)

BHLM Bouchard, Hernández Serrano, Liu, Mulase. *Mirror symmetry for orbifold Hurwitz numbers* (2013)

DoDM Do, Dyer, Mathews. *Topological recursion and a quantum curve for monotone Hurwitz numbers* (2014)

DoLN Do, Leigh, Norbury. *Orbifold Hurwitz numbers and Eynard–Orantin invariants* (2012)

DoM Do, Manescu. *Quantum curves for the enumeration of ribbon graphs and hypermaps* (2013)

DoN Do, Norbury. *Notes on the Kazarian–Zograf dessin count* (2014)

DM Dumitrescu, Mulase. *Quantum curves for Hitchin vibrations and the Eynard–Orantin theory* (2013)

DMNPS Dunin–Barkowski, Mulase, Norbury, Popolitov, Shadrin. *Quantum spectral curve for the Gromov–Witten theory of the complex projective line* (2013)

DMSS Dumitrescu, Mulase, Safnuk, Sorkin. *The spectral curve of the Eynard–Orantin recursion via the Laplace transform* (2012)

DOSS Dunin–Barkowski, Orantin, Shadrin, Spitz. *Identification of the Givental formula with the spectral curve topological recursion procedure* (2012)

EMS Eynard, Mulase, Safnuk. *The Laplace transform of the cut-and-join equation and the Bouchard–Mariño conjecture on Hurwitz numbers* (2009)

EO Eynard, Orantin. *Topological recursion in enumerative geometry and random matrices* (2008)
Computation of open Gromov–Witten invariants for toric Calabi–Yau 3-folds by topological recursion (2012)

FLZ Fang, Liu, Zong. *All-genus open-closed mirror symmetry for affine toric Calabi–Yau 3-orbifolds* (2013)

KZ Kazarian, Zograf. *Virasoro constraints and topological recursion for Grothendieck's dessin counting* (2014)

MS Mulase, Sułkowski. *Spectral curves and the Schrödinger equations for the Eynard–Orantin recursion* (2012)

MSS Mulase, Shadrin, Spitz. *The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures* (2013)

N Norbury. *String and dilation equations for counting lattice points in the moduli space of curves* (2009)

Z Zhou. *Intersection numbers on Deligne–Mumford moduli spaces and quantum Airy curve* (2012)
Quantum mirror curves for $\mathbb{CP}^3$ *and the resolved conifold* (2012)